J. Fluid Mech. (1970), *vol.* 44, *part* 1, *pp.* 19–31
Printed in Great Britain

A note on Hamilton's principle for perfect fluids

By FRANCIS P. BRETHERTON
Department of Earth and Planetary Sciences, The Johns Hopkins University

(Received 24 September 1969 and in revised form 12 May 1970)

A derivation is given of the Eulerian equations of motion directly from the Lagrangian formulation of Hamilton's principle. The circulation round a circuit of material particles of uniform entropy appears as a constant of the motion associated with the indistinguishability of fluid elements with equal density, entropy and velocity. A discussion is given of the Lin constraint, and it is pointed out that, for a barotropic fluid, the variational principle recently suggested by Seliger & Whitham does not permit velocity fields in which the vortex lines are knotted.

1. Introduction

The equations of motion for the adiabatic flow of an inviscid compressible fluid subject to conservative body forces may be obtained from Hamilton's principle in several ways (Herivel 1955; Lin 1963; Serrin 1959; Eckart 1960; Penfield 1966; Seliger & Whitham 1968). The most direct approach yields a Lagrangian formulation of the equations of motion, and, although Eckart (1960) has shown how it provides an elegant treatment of the fundamental theorems of fluid dynamics, it has not been used to any great extent for further developments, largely because the Lagrangian specification of the flow is unnecessary and inappropriate for many problems. Herivel and Lin proposed a different form of the variational principle, in which no direct mention is made of the positions of individual material particles. This gives the instantaneous velocity field in terms of gradients of potentials, but, although it contains complete information about the dynamics of the system, it too is not always very convenient in applications, because the physical significance of the potentials themselves is not very clear, and there are substantial indeterminacies and redundancies in their definition.

The primary purpose of this note is to propose a hybrid approach, whereby the Eulerian form of the equations of motion is obtained directly without the introduction of potentials. It permits a new derivation of Kelvin's circulation theorem, which is seen to describe the constants of the motion associated with the indistinguishability of different fluid particles with the same velocity, density and entropy. Some discussion is also given of the precise relationship of the Herivel–Lin formulation to Hamilton's principle, in particular of the significance of the Lin constraint. None of the final results are new, but it is hoped that the treatment helps clarify some of the fundamentals of fluid dynamics. It arose in the course of investigations into the general properties of wave propagation, and its application there will be reported elsewhere.

2. Orientation

In this section we recapitulate briefly the essentials of the two main earlier approaches. These have been presented many times before (e.g. Serrin 1959), but back references to many of the equations will be required later.

In the usual formulation of Lagrange's equations for a mechanical system, the position of every massive particle is determined by the geometry when certain generalized co-ordinates are known (although this geometrical specification may depend explicitly on time). For an unbounded fluid, the simplest set of generalized co-ordinates (a triply infinite continuum) is provided by the functions $\mathbf{x}(\tilde{\mathbf{x}})$, where $\mathbf{x}$ is the position vector of a particle labelled by its initial position $\tilde{\mathbf{x}}$. The geometrical specification is thus almost trivial. As the motion proceeds, the values of the generalized co-ordinates change, and attention is concentrated on the trajectories

$$\mathbf{x} = \mathbf{X}(\tilde{\mathbf{x}}, t). \tag{1}$$

At each time t, the functions (1) induce a mapping (assumed one–one), of which the inverse is

$$\tilde{\mathbf{x}} = \tilde{\mathbf{X}}(\mathbf{x}, t). \tag{2}$$

The density $\rho(\mathbf{x}, t)$ is related to the initial density $\tilde{\rho}$ along the appropriate trajectory by the Jacobian of the mapping,

$$\rho = \tilde{\rho} \left\{ \frac{\partial(\mathbf{X})}{\partial(\tilde{\mathbf{x}})} \right\}^{-1}, \tag{3}$$

and, if the entropy s of a fluid particle is constant,

$$s = \tilde{s}(\tilde{\mathbf{x}}). \tag{4}$$

If the internal energy per unit mass is $E(\rho, s)$, and the potential energy is $\Phi(\mathbf{x})$, the action integral for an unbounded fluid is

$$A = \int dt \int \left\{ \frac{1}{2} \left| \frac{d\mathbf{X}}{dt} \right|^2 - E(\rho, s) - \Phi(\mathbf{X}) \right\} \tilde{\rho}\, d\tilde{v}, \tag{5}$$

where

$$\frac{d}{dt} \equiv \left(\frac{\partial}{\partial t} \right)_{\tilde{\mathbf{x}}=\text{constant}}.$$

The integration is over all the mass in the system (identified by the initial density $\tilde{\rho}$ and the initial element of volume $d\tilde{v}$), and then over all time, all variables being considered as functions of $\tilde{\mathbf{x}}$ (the label on each generalized co-ordinate) and t.

We now consider suitably differentiable infinitesimal variations in the particle trajectories, which vanish for t outside some limited range $[t_1, t_2]$, and for $\tilde{\mathbf{x}}$ outside some finite domain $\tilde{V}$. The restriction at $t = \pm\infty$ is an essential part of Hamilton's principle. These variations induce changes $\Delta\rho$ in the mass per unit volume associated with a given particle (or label $\tilde{\mathbf{x}}$). The latter may conveniently be expressed in the hybrid Eulerian–Lagrangian form,

$$\frac{1}{\rho} \Delta\rho = -\nabla . (\Delta \mathbf{X}), \tag{6}$$

where ∇ is the gradient operator $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ with respect to current position $\mathbf{x}$, and, for the purpose of differentiation, $\Delta\mathbf{X}$ is expressed by means of (2) as a function of $\mathbf{x}$, t. According to (4),

$$\Delta s = 0. \tag{7}$$

Substituting into the action integral (5), integrating by parts and setting $\delta A = 0$ for all permissible $\Delta\mathbf{X}$, we obtain after some manipulation the Lagrangian equations of motion,

$$\rho \frac{d^2\mathbf{X}}{dt^2} + \rho\nabla\Phi + \nabla p = 0, \tag{8}$$

where

$$p = \rho^2 \left(\frac{\partial E}{\partial \rho}\right)_s. \tag{9}$$

In an Eulerian representation, on the other hand, the system is described by the velocity $\mathbf{u}(\mathbf{x}, t)$ at a fixed point, regardless of which fluid particle is actually there. It was shown by Herivel (1955) and Lin (1963) that the equations of motion may be derived from the variational principle,

$$\delta \int \left[\rho\{\tfrac{1}{2}|\mathbf{u}|^2 - E(\rho, s) - \Phi(\mathbf{x})\} + \alpha\left\{\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{u})\right\} + \rho\beta\frac{ds}{dt} + \sum_{i=1}^{3} \rho\gamma_i \frac{d\lambda_i}{dt}\right] dv\, dt = 0, \tag{10}$$

where $\mathbf{u}, \rho, s$ are regarded as functions of the current Cartesian co-ordinates $\mathbf{x}$ and

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}.\nabla.$$

The permitted variations in $\mathbf{u}$, ρ, s, α, β, γ_i, λ_i $(i = 1, 2, 3)$ are independent, continuously differentiable and vanish for $|\mathbf{x}|$ or $|t|$ sufficiently large. The Lagrangian multipliers α, β ensure conservation of mass and entropy, respectively

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{u}) = 0, \quad \frac{ds}{dt} = 0. \tag{11}$$

The term

$$\sum_{i=1}^{3} \gamma_i \frac{d\lambda_i}{dt}$$

was introduced by Lin. If it is omitted, the variational principle permits only a restricted class of velocity fields $\mathbf{u}$. Making variations $\delta\mathbf{u}$, we see that

$$\mathbf{u} = \nabla\alpha - \beta\nabla s - \sum_i \gamma_i \nabla\lambda_i, \tag{12}$$

so that $\mathbf{u}$ is described by the 8 potentials α, β, γ_i, λ_i. Without the last term, so that only α and β are included, it is easy to see that, for any closed circuit Γ lying in a surface of constant entropy, the circulation $\oint_\Gamma \mathbf{u}\,.\,d\mathbf{l}$ vanishes identically, and the velocity field $\mathbf{u}$ is not completely general. However, with the Lin constraint there are 6 additional potentials, which provide a more than adequate number of degrees of freedom (cf. Seliger & Whitham 1968). Because a number of misstatements about it have appeared in the literature, the physical significance of the Lin constraint is discussed in § 5.

Meanwhile, it should be noticed that the connexion between the variational principle (10) and Hamilton's principle is not entirely obvious. The state of the system is determined by the variables $\mathbf{u}$, ρ, s, etc., which do not specify the position of any individual material particle. Indeed, for a given velocity field, particle trajectories may only be obtained by an additional integration, of

$$\frac{d\mathbf{X}}{dt} = \mathbf{u}(\mathbf{X}, t), \tag{13}$$

so that the connexion between the geometrical configuration and the instantaneous values of the generalized co-ordinates is not unique. Besides equations (11) and (12), the Euler–Lagrange equations for the principle determine the time derivatives of the potentials, i.e.

$$\frac{d\alpha}{dt} = \tfrac{1}{2}|\mathbf{u}|^2 - E - \Phi - \frac{p}{\rho}, \tag{14}$$

$$\frac{d\beta}{dt} = -\frac{\partial E}{\partial s}, \tag{15}$$

$$\frac{d\lambda_i}{dt} = 0, \tag{16}$$

$$\frac{\partial}{\partial t}(\rho\gamma_i) + \nabla . (\rho\gamma_i \mathbf{u}) = 0, \tag{17}$$

and some elimination is required before the conventional momentum equation is recovered. The specification (12) of the velocity field in terms of potentials is cumbersome, and for some applications is distinctly inconvenient. Although the number of potentials necessary may be reduced from 8 to 4 (see § 6), for a given $\mathbf{u}(\mathbf{x})$, $\rho(\mathbf{x})$, $s(\mathbf{x})$ considerable arbitrariness remains in their definitions, and their physical significance is obscure. In particular, even in a state of equilibrium at rest, α and β decrease monotonically with time, and the λ_i and γ_i could be any functions of position for which

$$\sum_i \nabla\gamma_i \times \nabla\lambda_i + \nabla\beta \times \nabla s = 0. \tag{18}$$

3. Derivation of the Eulerian form of the momentum equation

In the Lagrangian formulation ((1)–(9)), all variables were functions of the initial co-ordinates $\tilde{\mathbf{x}}$ and time. However, for a given set of particle trajectories (1), they can be expressed via the inverse mapping (2) as functions of the current co-ordinates $\mathbf{x}$, t. The element of mass $\tilde{\rho}\, d\tilde{v}$ becomes identically $\rho\, dv$ and the action integral (5) is

$$A = \int \{\tfrac{1}{2}|\mathbf{u}|^2 - E(\rho, s) - \Phi(\mathbf{x})\}\rho\, dv\, dt. \tag{19}$$

However, we must now distinguish between variations Δ incurred on a given label or particle $\tilde{\mathbf{x}}$ and variations δ at a point fixed in space. If infinitesimal changes $\Delta\mathbf{X}(\tilde{\mathbf{x}}, t)$ are made in the trajectories, corresponding variations

$$\delta\tilde{\mathbf{X}}(\mathbf{x}) = -\left(\Delta\mathbf{X} . \frac{\partial}{\partial\mathbf{x}}\right)\tilde{\mathbf{X}} \tag{20}$$

are implied in the inverse mapping, and, for any quantity ϕ,

$$\begin{aligned}\delta\phi &= \Delta\phi + \delta\tilde{\mathbf{X}}.\frac{\partial\phi}{\partial\tilde{\mathbf{x}}}\\ &= \Delta\phi - \Delta\mathbf{X}.\frac{\partial\tilde{\mathbf{X}}}{\partial\mathbf{x}}.\frac{\partial\phi}{\partial\tilde{\mathbf{x}}}\\ &= \Delta\phi - \Delta\mathbf{X}.\nabla\phi. \end{aligned} \tag{21}$$

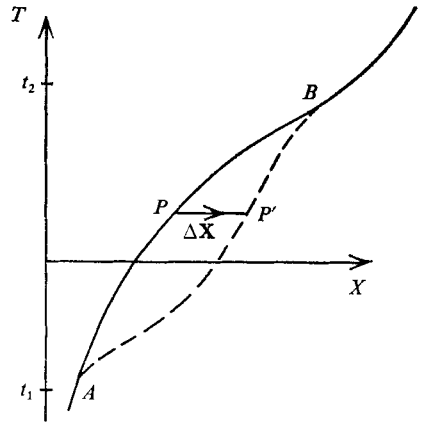

FIGURE 1. A particle trajectory in space-time APB, and a variant of it $AP'B$.

The essential point is indicated schematically in figure 1. APB represents the actual trajectory of a fluid particle and $AP'B$ a (virtual) variant of it. As required by Hamilton's principle, the two coincide for time t greater than some value t_2 or less than t_1. Then

$$\Delta\phi = \phi'(P') - \phi(P),$$

whereas

$$\begin{aligned}\delta\phi &= \phi'(P) - \phi(P)\\ &= \phi'(P') - \phi(P) - \{\phi'(P') - \phi'(P)\}\\ &\sim \Delta\phi - \overrightarrow{PP'}.\nabla\phi'\\ &\sim \Delta\phi - \Delta\mathbf{X}.\nabla\phi. \end{aligned}$$

As particular cases we have

$$\delta\rho = -\rho\nabla.(\Delta\mathbf{X}) - \Delta\mathbf{X}.\nabla\rho, \tag{22}$$

$$\delta s = 0 - \Delta\mathbf{X}.\nabla s, \tag{23}$$

$$\delta\mathbf{u} = \frac{d}{dt}(\Delta\mathbf{X}) - \Delta\mathbf{X}.\nabla\mathbf{u}. \tag{24}$$

Equations (22)–(24) express conservation of mass, constancy of entropy following a fluid particle, and the definition of $\mathbf{u}$ as the slope in space-time of the particle trajectory, relating variations to the displacements $\Delta\mathbf{X}$ only. For the purpose of calculation $\Delta\mathbf{X}$ itself as well as the other variables may be regarded as a function of $\mathbf{x}$, t. The important thing is that it should exist and should vanish for all sufficiently large $|\mathbf{x}|$ or $|t|$.

Substituting variations (22)–(24) into the action integral (19),

$$\begin{aligned}\delta A &= \int\Big[\Big\{\tfrac{1}{2}|\mathbf{u}|^2-\frac{\partial}{\partial\rho}(\rho E)_s-\Phi\Big\}\Delta\rho+\rho\mathbf{u}.\Delta\mathbf{u}+\rho\frac{\partial E}{\partial s}\Delta s \\ &\qquad -\rho\nabla\Phi.\Delta\mathbf{X}-\Delta\mathbf{X}.\nabla\{\tfrac{1}{2}\rho|\mathbf{u}|^2-\rho E-\rho\Phi\}\Big]\,dv\,dt \\ &= \int\Big[\nabla\Big\{\tfrac{1}{2}\rho|\mathbf{u}|^2-\rho\frac{\partial}{\partial\rho}(\rho E)-\rho\Phi\Big\}-\Big\{\frac{\partial}{\partial t}(\rho\mathbf{u})+\nabla.(\rho\mathbf{u}\mathbf{u})\Big\} \\ &\qquad -\rho\nabla\Phi-\nabla\{\tfrac{1}{2}\rho|\mathbf{u}|^2-\rho E-\rho\Phi\}\Big].\Delta\mathbf{X}\,dv\,dt \\ &= -\int\Big[\frac{\partial}{\partial t}(\rho\mathbf{u})+\nabla.(\rho\mathbf{u}\mathbf{u})+\rho\nabla\Phi+\nabla p\Big].\Delta\mathbf{X}\,dv\,dt, \end{aligned} \tag{25}$$

where p is defined by (9). This is true for arbitrary variations $\Delta\mathbf{X}$ which vanish at infinity, so Hamilton's principle asserts that

$$\frac{\partial}{\partial t}(\rho\mathbf{u})+\nabla.(\rho\mathbf{u}\mathbf{u})+\rho\nabla\Phi+\nabla p=0. \tag{26}$$

The Eulerian momentum equation is thus obtained as the variational derivative of the total action with respect to localized particle displacements.

We observe also that if in the identities (22)–(24) we let $\Delta\mathbf{X}=\mathbf{u}\,\delta t$, where δt is an infinitesimal constant, the time development of the system is delayed by an amount δt. We obtain expressions for the time derivatives at a point, associated with the actual motion:

$$\frac{\partial\rho}{\partial t}=-\rho(\nabla.\mathbf{u})-\mathbf{u}.\nabla\rho, \tag{27}$$

$$\frac{\partial s}{\partial t}=-\mathbf{u}.\nabla s, \tag{28}$$

and

$$\frac{\partial\mathbf{u}}{\partial t}=\frac{d}{dt}\mathbf{u}\qquad -(\mathbf{u}.\nabla)\mathbf{u}. \tag{29}$$

In this formulation these appear as identities, built explicitly into the geometrical specification of the system by (3) and (13). As Herivel showed, conditions (3) and (4) could be adjoined to the basic variational principle by Lagrange multipliers, but this introduces additional variables and the directness of the derivation is lost.

4. Kelvin's circulation theorem

In a homogeneous fluid, elements with the same velocity, density and entropy are indistinguishable, and they may be interchanged without affecting the physically interesting properties of the system at all. It is because of this indistinguishability that the Eulerian representation in terms of $\mathbf{u}$, ρ and s is more useful than the Lagrangian one in terms of particle positions. The latter carries redundant information which is not normally significant in the dynamics of the flow (though it is important for a correct application of Hamilton's principle). Associated with the invariance of the action integral (19) under a reshuffling of particles which leaves the fields $\mathbf{u}(\mathbf{x},t)$, $\rho(\mathbf{x},t)$ and $s(\mathbf{x},t)$ unaltered, we must

expect some fundamental constants of the motion. As we will now see, they are just those implied by Kelvin's circulation theorem.

Consider an instantaneous displacement $\Delta\mathbf{X}_1(\mathbf{x})$ under which the material particles in a closed solenoidal filament of infinitesimal cross-section S and centre-line Γ_1 are substituted for one another cyclically (figure 2), the mass Δm passing each cross-section of the filament being the same. Particles outside the filament

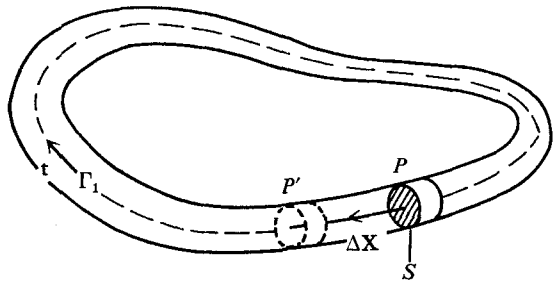

FIGURE 2. A substitution displacement of fluid particles around a filament centred on the circuit Γ_1. As the particle at P is displaced to P', its density is adjusted to that of the particle previously at P'.

are left undisturbed. This change could be envisaged as due to localized body forces acting during an infinitesimal interval about time $t = t_1$. The displacement $\Delta\mathbf{X}_1$ must be parallel to the local unit tangent vector $\mathbf{t}$ of the filament, and

$$\Delta m = \rho S\,|\Delta\mathbf{X}_1| = \text{constant}. \tag{30}$$

This condition ensures that the density distribution ρ after the displacement is the same as before ($\delta\rho = 0$). We suppose also that Γ_1 lies in a surface of constant s, so that the local change δs required by (23) vanishes. An instantaneous displacement at time t_1 implies a particle velocity which is a Dirac delta function of time:

$$\delta\mathbf{u} = \Delta\mathbf{X}_1(\mathbf{x})\,\delta(t-t_1); \tag{31}$$

and the associated variation in the total action is

$$\begin{aligned}\delta A_1 &= \int dt \int \left\{\rho\mathbf{u}.\delta\mathbf{u} + \left(\tfrac{1}{2}|\mathbf{u}|^2 - E - \phi - \rho\frac{\partial E}{\partial\rho}\right)\delta\rho - \rho\frac{\partial E}{\partial s}\,\delta s\right\} dv \\ &= \int \rho\mathbf{u}.\Delta\mathbf{X}_1\,dv \\ &= \int_{\Gamma_1} \mathbf{u}.\mathbf{t}\rho\,|\Delta\mathbf{X}_1|\,S\,dl \\ &= \Delta m \oint_{\Gamma_1} \mathbf{u}.d\mathbf{l}. \end{aligned} \tag{32}$$

Here

$$d\mathbf{l} = \mathbf{t}\,dl$$

is the line element along the filament Γ_1, so that the element of volume of the solenoid is $dv = S\,dl$. Previous and subsequent to the infinitesimal interval around $t = t_1$, we suppose that the velocity field $\mathbf{u}(\mathbf{x}, t)$ is unaltered, so that the

time development of all the fields according to the identities (27)–(29) is unaffected by the variation and, except for the contribution (32) already calculated, the variation in the total action vanishes.

However, in one important respect, this particle substitution does not meet the requirements of a variation under Hamilton's principle. Although the true velocity field $\mathbf{u}(\mathbf{x}, t)$ and its variant $\mathbf{u}'(\mathbf{x}, t)$ coincide for all times except close to $t = t_1$, so that

$$\delta\mathbf{u}(\mathbf{x}, \infty) = \delta\mathbf{u}(\mathbf{x}, -\infty) = 0, \tag{33}$$

the same is not true of the trajectories. A particle displaced at time t_1 to a neighbouring point in physical space will remain on the trajectory defined by (13) through that point. The displacement $\Delta\mathbf{X}(\mathbf{x}, t)$ changes for $t > t_1$ like the infinitesimal line element separating two material fluid particles, and does not vanish

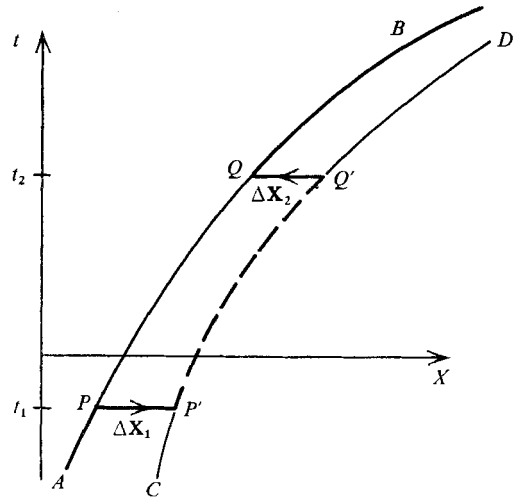

FIGURE 3. Particle trajectories $APQB$, $CP'Q'D$ for a velocity field $\mathbf{u}(\mathbf{x}, t)$, and a variant trajectory $APP'Q'QB$ subject to substitution displacements at t_1 and t_2.

as $t \to +\infty$. To restore it to zero a second substitution displacement is necessary at some time t_2, defined by a mass exchange around the solenoid Γ_2 consisting of the same material particles which were involved in Γ_1, of magnitude equal to Δm but in the opposite sense. The change in the total action under this second substitution displacement is

$$\delta A_2 = -\Delta m \oint_{\Gamma_2} \mathbf{u}.d\mathbf{l}. \tag{34}$$

Then, for times $t > t_2$, the variant trajectories coincide in all respects with the originals, and Hamilton's principle applies. It states that

$$\delta A_1 + \delta A_2 = 0,$$

i.e.

$$\oint_{\Gamma_1} \mathbf{u}.d\mathbf{l} = \oint_{\Gamma_2} \mathbf{u}.d\mathbf{l}. \tag{35}$$

Clearly the same is true at any times t_1, t_2, for any closed material filament consisting of particles all with the same entropy. We thus have precisely Kelvin's circulation theorem.

The instantaneous localized displacements just considered are not, of course, continuous and are not strictly permitted under the rules for variational principles. However, we may approximate them by infinitely differentiable functions as closely as we please. The density and entropy distributions will be unaltered by an instantaneous displacement field $\Delta\mathbf{X}(\mathbf{x})$ if

$$\nabla.(\rho\Delta\mathbf{X}) = 0, \tag{36}$$

$$\Delta\mathbf{X}.\nabla s = 0. \tag{37}$$

Locally, at least, the most general such field may be written

$$\rho\Delta\mathbf{X} = \alpha(s,\phi)\nabla s\times\nabla\phi, \tag{38}$$

where ϕ is an arbitrary function of position, and $\alpha(s,\phi)$ is also arbitrary. For (38) certainly implies (36) and (37). Conversely, consider the curves Γ with tangents everywhere parallel to a given $\Delta\mathbf{X}(\mathbf{x})$. By (37) these certainly lie in the surface s = constant. Also, let $\phi(\mathbf{x})$ be any other smooth function which is also constant along each Γ. Then $\Delta\mathbf{X}$ is everywhere parallel to the intersection of the surfaces s = constant and ϕ = constant, i.e.

$$\rho\Delta\mathbf{X} = \alpha(\mathbf{x})\nabla s\times\nabla\phi \tag{39}$$

for some α. Now (36) shows that

$$\nabla\alpha.(\nabla s\times\nabla\phi) = 0, \tag{40}$$

i.e. α is constant along each Γ, and may be written as a function of s and ϕ only. Now the infinitesimal cross-sectional area $\mathcal{S}\mathbf{t}$ of the filament contained between the neighbouring surfaces ϕ and ϕ', s and s', is given by

$$\mathcal{S}\mathbf{t} = \frac{\nabla s\times\nabla\phi}{|\nabla s\times\nabla\phi|^2}(S'-s)(\phi'-\phi), \tag{41}$$

so the mass displaced along it is simply

$$\rho\Delta\mathbf{X}.\mathbf{t}\mathcal{S} = \alpha(s,\phi)(s'-s)(\phi'-\phi). \tag{42}$$

A sequence of such continuous functions $\alpha(s,\phi)$ may be used to define a Dirac delta function of given strength Δm, centred on a given circuit Γ' defined by s', ϕ'

$$\alpha(s,\phi) = \Delta m\,\delta(s-s',\phi-\phi'). \tag{43}$$

This is the displacement $\Delta\mathbf{X}_1$ considered above (with the s', ϕ' replacing s, ϕ). Between times t_1 and t_2, s' and ϕ' are Lagrangian variables, which are constant following fluid particles. At time t_2 the displacement defined by (43) is repeated, but with a minus sign in front of Δm.

However, this analytical characterization of the displacements described in figure 2 suffers from a substantial defect. Although a representation of the form (39) can always be found locally, it is not obvious that ϕ will be single valued over the *whole* three-dimensional Euclidean space. Whether this is so for a given displacement field $\Delta\mathbf{X}$ apparently depends on the topology of the whole set of tangent curves Γ. For example, if these have a shape similar to the strands in a rope spliced to form a continuous loop, ϕ clearly must be singular somewhere. This point deserves further investigation.

5. The Lin constraint

The present approach helps clarify the significance of the Lin constraint. The effect of the latter is not (as stated by Seliger & Whitham 1968) to ensure that the particle trajectories $\mathbf{X}(\tilde{\mathbf{x}}, t)$ exist, because (13) has solutions for any continuous $\mathbf{u}(\mathbf{x}, t)$, and these may always be identified by the initial co-ordinates $\tilde{\mathbf{x}}$. Neither does it primarily require that the intersection of such trajectories with the physical boundaries of the fluid should remain fixed.

Rather, as apparently realized by Lin himself and Eckart (1960), the introduction of the functions λ_i serves to ensure that no variation $\Delta\mathbf{X}$ in the trajectories persists as $t \to \infty$. It is worth considering in a little more detail the precise relationship of the variational principle (10) to Hamilton's principle, and how the restriction of the trajectories is actually achieved.

The variational principle is expressed in terms of the variable functions $\mathbf{u}$, ρ, s, α, β, γ_i, λ_i, which serve as generalized co-ordinates. No mention is made of how the positions of the massive particles in the system are related to the generalized co-ordinates, so the connexion with the elementary formulations of a Lagrangian and of Hamilton's principle is not immediately obvious. Although, for given $\mathbf{u}(\mathbf{x}, t)$, the solution curves $\mathbf{x} = \mathbf{X}(\tilde{\mathbf{x}}, t)$ of (13) may be uniquely labelled by the co-ordinates $\tilde{\mathbf{x}}$ corresponding to $t = 0$, these labels are not necessarily attached to particular material particles. However, variation of α, β and γ_i in (10) implies (11) and (16) for the conservation of mass, and that the entropy s and the λ_i ($i = 1, 2, 3$) are constants along these solution curves. These equations could, in principle, be integrated to yield (3) and (4), and a material particle could thus consistently be associated with each label $\tilde{\mathbf{x}}$. Something approaching the required geometrical connexion is then implicit, and the integral in (10) reduces to the total action (19).

However, statements (11) and (16) are inferences from the variational principle, and must be treated with caution if they are to be used to modify the formulation of the principle itself. Nevertheless, such use is permitted if subsequent variations in the remaining variables are restricted in such a way as never to violate the assumed statements. Although they are not all independent, it is unnecessary to eliminate explicitly between them. The variational principle thus obtained from (10) by ignoring terms in α, β and γ_i, but making the most general variations $\delta\mathbf{u}$, $\delta\rho$, δs, $\delta\lambda_i$ consistent with (11) and (16), is then equivalent to (10) itself with the fields α, β, γ_i eliminated after all variations have been made.

An explicit statement of the restrictions on $\delta\mathbf{u}$, $\delta\rho$ and δs has already been given, namely (22)–(24). To see this, consider the trajectory $\mathbf{X}(\tilde{\mathbf{x}}, t)$ defined by (13) for a field $\mathbf{u}(\mathbf{x}, t)$, and the corresponding $\mathbf{X}'(\tilde{\mathbf{x}}, t)$ for a variant $\mathbf{u}'(\mathbf{x}, t)$. For all $t < t_1$

$$\delta\mathbf{u} = \mathbf{u}' - \mathbf{u} = 0,$$

and we may choose

$$\Delta\mathbf{X} = \mathbf{X}'(\tilde{\mathbf{x}}, t) - \mathbf{X}(\tilde{\mathbf{x}}, t) = 0 \tag{44}$$

there, thus eliminating ambiguity in the original attachment of labels to material particles. Subsequently, $\Delta\mathbf{X}$ is the unique solution of (24) determined along the

characteristic trajectories $\mathbf{X}(\tilde{\mathbf{x}}, t)$, and a general $\delta\mathbf{u}$ implies a corresponding $\Delta\mathbf{X}$, and conversely. It may be verified that $\delta\rho$ given by (22) is then the solution of

$$\frac{\partial}{\partial t}(\delta\rho) + \nabla.(\delta\rho\mathbf{u}) + \nabla.(\rho\delta\mathbf{u}) = 0$$

which corresponds to $\delta\rho = 0$ when $t < t_1$. Similarly for (23). We must also adjoin

$$\delta\lambda_i = -\Delta\mathbf{X}.\nabla\lambda_i. \tag{45}$$

Thus, the variational principle (10) is indeed very close to the version of Hamilton's principle described in § 3. However, there is one important difference which must be clearly recognized.

As has already been emphasized, the condition $\delta\mathbf{u} = 0$ for $|t|$ sufficiently large, is *not* sufficient to ensure that the variations $\Delta\mathbf{X}$ inferred from equation (24) also vanish there, although the converse is of course true. The geometrical specification implied by (10) depends on the whole past history of the field $\mathbf{u}(\mathbf{x}, t)$, not on its instantaneous value. The restrictions

$$\delta\mathbf{u} = \delta\rho = \delta s = 0 \quad (t > t_2),$$

which are built into the statement of the variational principle (10), do not of themselves tie down the ends of the particle trajectories in the manner required by Hamilton's principle. The degree of freedom remaining corresponds precisely to an arbitrary substitution displacement as defined in § 4, and the vanishing of the corresponding variation in the total action implies that

$$\oint_{\Gamma_2} \mathbf{u}.d\mathbf{l} = 0 \tag{46}$$

for every closed circuit Γ_2 lying in a surface of constant entropy. This limitation on the velocity fields $\mathbf{u}$ is just that inherent in Herivel's original formulation of the variational principle.

This difficulty is overcome by the inclusion of the terms $\gamma_i(d\lambda_i/dt)$. Variation of the Lagrange multipliers γ_i implies that the λ_i are consistent along the particle trajectories, and the λ_i could be chosen as the original Cartesian co-ordinates $\tilde{\mathbf{x}}_i$ $(i = 1, 2, 3)$ of the trajectory. However, any set of differentiable functions, for which $\partial(\lambda_1, \lambda_2, \lambda_3)/\partial(x_1, x_2, x_3) \neq 0$ or ∞ anywhere, would do as well. Since the λ_i are constants of the motion, the mapping $\boldsymbol{\lambda} \leftrightarrow \mathbf{x}$ remains non-singular for all time if it was so originally. The crucial part of the Lin constraint is the assumption for (10) that

$$\delta\lambda_i = 0, \quad t < t_1 \quad \text{and} \quad t > t_2. \tag{47}$$

This completely removes the indeterminacy in $\Delta\mathbf{X}$, because according to equations (45), (47), $\Delta\mathbf{X}$ is now perpendicular to three non-coplanar vectors $\nabla\lambda_i$ $(i = 1, 2, 3)$.

Thus, if variations in (10) are accomplished in two stages (first the Lagrange multipliers α, β, γ_i to establish the geometrical framework, and then the remaining variables $\mathbf{u}, \rho, s, \lambda_i$ subject to the geometrical constraints), the correspondence with Hamilton's principle is exact. It should be noted that, once they have been varied, all reference to α, β, γ_i is suppressed. Any inferences about

them would involve variations $\delta\mathbf{u}$, etc., which violate the geometrical correspondence. Thus, from this viewpoint, the existence of a representation of the velocity of the form (12) for some single-valued functions α, β, γ_i satisfying (14), (15) and (17) must be inferred from the equations of motion, rather than conversely.

6. Reduction in the number of potentials

According to Seliger & Whitham (1968), we may use the variational principle (10) with only one Lin variable λ and its Lagrange multiplier γ, without impairing the generality of the states of the physical system which are described. This reduces the number of potentials from 8 $(\alpha, \beta, \gamma_i, \lambda_i)$ to 4 $(\alpha, \beta, \gamma, \lambda)$, and is a real economy. However, their remark requires qualification.

Consider first a barotropic fluid, in which ∇s vanishes everywhere. We have then the velocity field described by Clebsch co-ordinates,

$$\mathbf{u} = \nabla\alpha - \gamma\nabla\lambda. \tag{48}$$

Now, any vector field $\mathbf{u}$ may be described *locally* in terms of three functions of position in this way, but, if α, γ and λ are to be single-valued, there are some integral constraints on $\mathbf{u}$, which may seriously reduce the value of the representation.† Thus, the vorticity is given by

$$\boldsymbol{\zeta} = \nabla \times \mathbf{u} = -\nabla y \times \nabla\lambda, \tag{49}$$

and, for any volume V bounded by the simple closed surface Σ,

$$\begin{aligned}\int_V \boldsymbol{\zeta}.\mathbf{u}\, dv &= -\int_V \{\nabla\alpha.\nabla\gamma\times\nabla\lambda\}\, dv \\ &= -\int_V \nabla.\{\alpha\nabla\gamma\times\nabla\lambda\}\, dv \\ &= \int_\Sigma \alpha\boldsymbol{\zeta}.d\boldsymbol{\Sigma}.\end{aligned} \tag{50}$$

If Σ is chosen such that $\boldsymbol{\zeta}.d\boldsymbol{\Sigma} = 0$ everywhere on it, i.e. if V is made up of the superposition of a number of closed vortex filaments, we must have

$$\int_V \boldsymbol{\zeta}.\mathbf{u}\, dv = 0. \tag{51}$$

For a general velocity field $\mathbf{u}$, $\int_V \boldsymbol{\zeta}.\mathbf{u}\, dv$ is a measure of the knottedness of the vortex lines (Moffatt 1969), and does not in general vanish. Indeed, if the volume V moves with the fluid particles contained in it, this integral is a constant of the motion. For an individual closed solenoidal vortex filament of infinitesimal local cross-sectional area S, the strength

$$C = |\boldsymbol{\zeta}|\, S \tag{52}$$

† The author is grateful to Dr H. K. Moffatt for pointing this out.

is a constant around the filament and with time. But

$$\int_V \boldsymbol{\zeta}.\mathbf{u}\,dv = \int |\boldsymbol{\zeta}|\, S\mathbf{u}.\mathbf{t}\,dl$$
$$= C \oint \mathbf{u}.d\mathbf{l}, \tag{53}$$

which is constant by Kelvin's circulation theorem. Clearly, the same is true for the superposition of any number of vortex filaments.

Thus, if the initial conditions are such that the vortex lines are knotted, the potentials in the Clebsch representation are essentially multiple valued, and must have singularities somewhere. Thus, any conclusions drawn from the variational principle concerning integrals over finite regions of the velocity field must be regarded with reserve, until the adequacy of the representation has been demonstrated. The equations of motion themselves require only *local* validity in a neighbourhood of the point under consideration, and are unaffected.

When $\nabla s \neq 0$, however, an additional degree of freedom is available, corresponding to the potential β. Although the author has been unable to prove it, it appears likely that single-valued functions can be found for any velocity field $\mathbf{u}$, and the reduction to 4 potentials is free from objection. The relation of the corresponding variational principle to Hamilton's principle remains obscure.

The author is indebted to Dr W. L. Siegmann for reading the manuscript, and for invaluable discussions of its contents. This research was supported by the National Science Foundation, Grant GA16603.

REFERENCES

ECKART, C. 1960 *Phys. Fluids*, **3**, 421.

HERIVEL, J. W. 1955 *Proc. Camb. Phil. Soc.* **51**, 344.

LIN, C. C. 1963 *Liquidhelium, Proc. Int. School of Physics, Course XXI.* New York: Academic.

MOFFATT, H. K. 1969 *J. Fluid Mech.* **35**, 117.

PENFIELD, P. 1966 *Phys. Fluids*, **9**, 1184.

SELIGER, R. L. & WHITHAM, G. B. 1968 *Proc. Roy. Soc.* A **305**, 1.

SERRIN, J. 1959 *Handbuch der Physik*, vol. 8, p. 125.